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Modifying the gravity: successes and failures

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Gravity and Cosmology

- **The Einstein theory of gravity is extremely successful. And it has provided us with a breath-taking possibility to scientifically discuss the origins of the whole Universe and its history.**

But: 1. Dark Matter is somewhat intriguing

2. Dark Energy points at a technically very unnatural value of the cosmological constant

3. We need a period of inflation in the very early Universe

4. The underlying fundamental physics of inflation is not known

5. Signatures of statistical anisotropy in the CMB radiation, and the lack of correlations at the largest angular scales

Modifying the gravity?

- On cosmological scales we have some problems with general relativity, therefore people are looking for IR modifications
- One of the most straight-forward IR modifications amounts to introducing a mass term – automatically introduces an auxiliary metric which can be made dynamical
- However it is not that easy in GR (ghosts etc)
- Only very recently a potentially healthy modification has been obtained: de Rham – Gabadadze – Tolley non-linear massive gravity
- There are still some problems despite the healthy number (five) of degrees of freedom: viable cosmological regimes are not easy to obtain, superluminal propagation is ubiquitous (Deser, Waldron)
- Nevertheless, it is very remarkable that a reasonable candidate is at hand; and the structure of the model is worth studying
- Other possible modifications are also of great interest

Mimetic Dark Matter

Recently, Chamseddine and Mukhanov have proposed a new model in arXiv: 1308.5410

They use the Einstein-Hilbert action with the physical metric given

$$g_{\mu\nu} = \left(\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\tilde{g}_{\mu\nu}$$

in terms of an auxiliary metric and a scalar field.

Equations of motion: $(G^{\mu\nu} - T^{\mu\nu}\ ) - (G - T\ )\, g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi = 0,$

$$\nabla_\kappa\left((G - T\ )\,\partial^\kappa\phi\right) = 0$$

However, directly from the definition follows a constraint

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 1$$

The trace part of Einstein equations is satisfied identically – more freedom which looks like a pressureless dust

$$\tilde{T}^{\mu\nu} = (G - T\ )\, g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\partial_\beta\phi$$

with $\varepsilon \equiv G - T, \quad u^\mu \equiv g^{\mu\alpha}\partial_\alpha\phi$ in $T^{\mu\nu} = (\varepsilon + p)\, u^\mu u^\nu - p g^{\mu\nu}$

On variational principle in Mimetic Dark Matter

(A. Golovnev; arXiv:1310.2790)

How could it have happened without changing the action?
The determinant is being varied in terms of the factor

$$\Omega(x) \equiv \tilde{g}^{\alpha\beta}(\partial_\alpha \phi)(\partial_\beta \phi)$$

Assume, for simplicity, the spatial homogeneity of the fields
In which class of functions do we vary? $\delta\phi = 0$ at $t = t_{in}$ and $t = t_{fin}$
It amounts to $\int\limits_{t_{in}}^{t_{fin}} dt\sqrt{\Omega} = t_{fin} - t_{in}$ **for a legitimate variation**
The narrower class means more general equations.

This is nothing more but a general problem of making derivative substitutions into the action. Suppose we have an action $S = \int \dot{x}^2 dt$, and substitute $x \equiv \dot{y}$ into it. After that, the equations of motion are of higher order, even for $x(t)$, and therefore one needs more Cauchy data. The reason is the same as above. We vary in the class of functions which is defined not only by vanishing of δx at the boundary but also by an extra $\int\limits_{t_{in}}^{t_{fin}} \delta x(t) dt = 0$ condition.

Compare to the cosmological constant in unimodular gravity!

Equivalent formulation of Mimetic Dark Matter

(A. Golovnev; arXiv:1310.2790)

Let's try another action

$$S = -\int \left(R(g) + \lambda^{\mu\nu}\left(g_{\mu\nu} - \tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi)\right)\right)\sqrt{-g}d^4x.$$

Varying with respect to various variables we get

$$g_{\mu\nu} = \left(\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\tilde{g}_{\mu\nu} \qquad \nabla_\mu\left(\lambda\partial^\mu\phi\right) = 0 \qquad G_{\mu\nu} + \lambda_{\mu\nu} = 0 \qquad \lambda_{\mu\nu} = \lambda(\partial_\mu\phi)(\partial_\nu\phi)$$

Finally, everything is equivalent to the initial formulation.

We can do even better!

The Lagrange multiplier is determined only by its trace part

So, let's try the following:

$$S = -\int \left(R(g) + \lambda\left(1 - g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi)\right)\right)\sqrt{-g}d^4x$$

No auxiliary metric!

But still equivalent to the Einstein equation

$$G_{\mu\nu} + \lambda(\partial_\mu\phi)(\partial_\nu\phi) = 0$$

Linearised GR

We use the "mostly plus" (-,+,+,+) sign convention, and the Einstein-Hilbert action. At quadratic level around Minkowski, the action reads

$$\sqrt{-g}\,R \approx -\frac{1}{4}\Big((\partial_\alpha h_{\mu\nu})(\partial^\alpha h^{\mu\nu}) - 2(\partial^\alpha h_{\mu\nu})(\partial^\nu h^\mu_\alpha) + 2(\partial_\alpha h^{\alpha\mu})(\partial_\mu h^\beta_\beta) - (\partial_\mu h^\alpha_\alpha)(\partial^\mu h^\beta_\beta)\Big)$$

In the usual variables $h_{00} = 2\phi$, $h_{0i} = \partial_i b + s_i$ where $\partial_i s_i \equiv 0$,
and $h_{ik} = 2\psi\,\delta_{ik} + 2\partial^2_{ik}\sigma + \partial_i v_k + \partial_k v_i + h^{(TT)}_{ik}$ where $\partial_i v_i \equiv 0$ $,\partial_i h^{(TT)}_{ik} \equiv 0$
and $h^{(TT)}_{ii} \equiv 0$ (and, of course, $g_{\mu\nu} \equiv \eta_{\mu\nu} + h_{\mu\nu}$), we get up to surface terms

$$-\frac{1}{4}(\partial_\alpha h^{(TT)}_{ik})(\partial^\alpha h^{(TT)}_{ik}) + \frac{1}{2}\left(\partial_k\left(\dot{v}_i - s_i\right)\right)^2 - 3\dot{\psi}^2 + (\partial_i\psi)^2 + 4\psi\,\Delta\left(\phi - \dot{b} + \ddot{\sigma}\right)$$

The transverse-traceless tensor sector is physical (gravitons, two degrees of freedom), there are four Lagrange multipliers (two scalars and one transverse vector), and the constraints are actually first class making four more variables unphysical (pure gauge).

We see the wrong sign kinetic term for ψ , however everything apart from $h^{(TT)}$ is not genuinely dynamical due to the gauge symmetry. But once we break it, we are to expect some problems to come about!

And indeed, after breaking the four gauge invariances, generically we get six degrees of freedom, five of the normal spin-2 field, and one of the additional scalar which is actually the Boulware-Deser ghost.

Fierz-Pauli action

- In massive gravity, we always need an auxiliary metric, even if a mere Minkowski one. This is just because otherwise there exist no non-derivative invariant to construct the potential from.
- Generically, a theory of massive gravity has six degrees of freedom: five of massive spin-two, and one of a scalar ghost
- Linearly, there is the Fierz-Pauli mass term with only five of them:

$$\frac{m^2}{4}\left(h_{\mu\nu}h^{\mu\nu}-h^{\mu}_{\mu}h^{\nu}_{\nu}\right)$$

- In Stueckelberg trick this is the action with Maxwellian kinetic term for the vector. Therefore, the scalar is healthy.
- The linear level Stueckelberg $h_{\mu\nu}\rightarrow h_{\mu\nu}+\partial_\mu\xi_\nu+\partial_\nu\xi_\mu$ $\xi_\mu\rightarrow\xi_\mu+\partial_\mu\pi$
- There is a well-known problem of vDVZ-discontinuity. It might very well be cured by Vainstein mechanism. But, anyway, non-linearly (or, equivalently, around a curved background) it develops the Boulware-Deser ghost (the sixth degree of freedom), be there any Vainstein mechanism or not, for generic non-linear completions of the potential.

dRGT-gravity

- It is an amusing fact that a family of completely BD-ghost-free massive gravity models does exist (Claudia de Rham, Gregory Gabadadze, Andrew Tolley). One can achieve that by successively eliminating the leading scalar self-interactions.
- dRGT have worked with the matrix H defined by

$$g^{-1}\eta \equiv I - H \qquad \text{and} \qquad K \equiv I - \sqrt{I-H}$$

and used the Stueckelberg trick in perturbation theory. The EFT cutoff is raised to $\Lambda_3 \equiv (m^2 M_{Pl})^{1/3}$ from the Vainshteinian $\Lambda_5 \equiv (m^4 M_{Pl})^{1/5}$ due to elimination of leading (kinetic) self-interactions of the scalar.

- The potential can be put into the form $V = 2\mathrm{m}^2\left(\left(\sqrt{g^{-1}\eta}\right)^{\mu}_{\mu} - 3\right)$
- It is just the first symmetric polynomial (trace) of the eigenvalues of the basic matrix $\sqrt{g^{-1}\eta}$
- Non-perturbatively, it works if a real matrix square root exists. This is not always the case, and there is also uniqueness problem.

$$\left(\frac{1}{5}\begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}\right)^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The generalisations

- The other possible mass terms are the second and the third order symmetric polynomials of the eigenvalues (the fourth one is just the determinant)

$$\mathrm{Tr}\sqrt{g^{-1}f}$$

$$\left(\mathrm{Tr}\sqrt{g^{-1}f}\right)^2-\mathrm{Tr}\left(\sqrt{g^{-1}f}\right)^2$$

$$\left(\mathrm{Tr}\sqrt{g^{-1}f}\right)^3-3\left(\mathrm{Tr}\sqrt{g^{-1}f}\right)\mathrm{Tr}\left(\sqrt{g^{-1}f}\right)^2+2\mathrm{Tr}\left(\sqrt{g^{-1}f}\right)^3$$

- One can use an arbitrary background metric $f_{\mu\nu}$ instead of the Minkowski one
- Generalisation to bimetric version is fairly straightforward since the symmetric polynomials of $\sqrt{g^{-1}f}$ multiplied by $\sqrt{-g}$ can be expressed in terms of those for $\sqrt{f^{-1}g}$ multiplied by $\sqrt{-f}$
- The theory can also be generalised by allowing the mass to depend on a scalar field – improvements in cosmological model building.

The ADM formalism

- The standard way to perform a non-perturbative analysis in gravity is to invoke the ADM decomposition

$$ds^2 \equiv -(N^2 - N_k N^k)\,dt^2 + 2N_i\,dx^i\,dt + \gamma_{ij}\,dx^i\,dx^j$$

- In standard GR the lapse and shift are Lagrange multipliers, and the associated constraints decrease the number of degrees of freedom in the γ from 6 to 2.

$$H = -\int d^3x\,\sqrt{\gamma}\left(N\left(R^{(3)} + \frac{1}{\gamma}\left(\frac{1}{2}\left(\pi_j^{\,j}\right)^2 - \pi_{ik}\pi^{ik}\right)\right) + 2N_i\nabla_k^{(3)}\pi^{ik}\right)$$

- In massive gravity with potential V, the Hamiltonian is non-linear in them, thus leaving generically the full set of **6**

$$H = -\int d^3x\,\sqrt{\gamma}\left(N\left(R^{(3)} + \frac{1}{\gamma}\left(\frac{1}{2}\left(\pi_j^{\,j}\right)^2 - \pi_{ik}\pi^{ik}\right) - V\right) + 2N_i\nabla_k^{(3)}\pi^{ik}\right)$$

where the indices are handled with respect to the metric γ, and π^{ik} are canonical momenta of γ_{ik}

Computing the Riemann components 1

We need the Riemann tensor (ADM) components. It is reasonable to calculate the connection components first. Anyway, they might be needed during the further steps. We will thoroughly trade velocities for the extrinsic curvatures using

$$K_{ij} = -N\Gamma^0_{ij} = \frac{1}{2N}\left(\overset{(3)}{\nabla}_i N_j + \overset{(3)}{\nabla}_j N_i - \dot{\gamma}_{ij}\right)$$

In what follows we need:

$$\begin{aligned}
\Gamma_{ij0} = \Gamma_{i0j} &= -NK_{ij} + \overset{(3)}{\nabla}_j N_i, \\
\Gamma_{ijk} &= \overset{(3)}{\Gamma}_{ijk}, \\
\Gamma^0_{00} &= \frac{1}{N}\left(\dot{N} + N^i\partial_i N - N^i N^j K_{ij}\right), \\
\Gamma^0_{0i} = \Gamma^0_{i0} &= \frac{1}{N}\left(\partial_i N - N^j K_{ij}\right), \\
\Gamma^i_{0j} = \Gamma^i_{j0} &= -\frac{N^i \partial_j N}{N} - N\left(\gamma^{ik} - \frac{N^i N^k}{N^2}\right)K_{kj} + \overset{(3)}{\nabla}_j N^i, \\
\Gamma^0_{ij} &= -\frac{1}{N}K_{ij}, \\
\Gamma^i_{jk} &= \overset{(3)}{\Gamma}{}^i_{jk} + \frac{N^i}{N}K_{jk}.
\end{aligned}$$

Computing the Riemann components 2

$$R_{ijkl} = g_{i\rho}\partial_k\Gamma^\rho_{lj} - g_{i\rho}\partial_l\Gamma^\rho_{kj} + \Gamma_{ik\rho}\Gamma^\rho_{lj} - \Gamma_{il\rho}\Gamma^\rho_{kj}$$

$$= -N_i\partial_k\left(\frac{1}{N}K_{jl}\right) + \gamma_{im}\partial_k\left(\overset{(3)}{\Gamma}{}^m_{jl} + \frac{N^m}{N}K_{jl}\right) - \frac{1}{N}K_{jl}\left(-NK_{ik} + \overset{(3)}{\nabla}_k N_i\right)$$

$$+ \overset{(3)}{\Gamma}_{ikm}\left(\overset{(3)}{\Gamma}{}^m_{lj} + \frac{N^m}{N}K_{lj}\right) - (k \leftrightarrow l)$$

$$= \overset{(3)}{R}_{ijkl} + K_{ik}K_{jl} - K_{il}K_{jk}$$

Then it would be easier to calculate

$$n^\mu \equiv \left(\tfrac{1}{N}, -\tfrac{N^i}{N}\right)$$

$$n_\mu \equiv \left(-N, \vec{0}\right)$$

$$n_\mu R^\mu{}_{i\alpha j} = n^\mu R_{\mu i\alpha j} = \frac{1}{N}R_{0i\alpha j} - \frac{N^k}{N}R_{ki\alpha j}$$

$$n_\mu R^\mu{}_{ijk} = -N\left(\partial_j\Gamma^0_{ki} + \Gamma^0_{j\rho}\Gamma^\rho_{ki}\right) + N\left(\partial_k\Gamma^0_{ji} + \Gamma^0_{k\rho}\Gamma^\rho_{ji}\right) = \partial_j K_{ki} + \overset{(3)}{\Gamma}{}^m_{ki}K_{jm} - (j \leftrightarrow k)$$

$$= \overset{(3)}{\nabla}_j K_{ki} - \overset{(3)}{\nabla}_k K_{ji}$$

Finally, after just a little bit of very simple algebra, we find

$$n_\mu R^\mu{}_{i0j} = \dot{K}_{ij} + \overset{(3)}{\nabla}_i\overset{(3)}{\nabla}_j N + NK_i{}^k K_{kj} - \overset{(3)}{\nabla}_j\left(K_{ik}N^k\right) - K_{kj}\overset{(3)}{\nabla}_i N^k.$$

we trivially transform it to a more symmetric form

$$n^\mu n^\nu R_{\mu i\nu j} = \frac{1}{N}\left(\dot{K}_{ij} + \overset{(3)}{\nabla}_i\overset{(3)}{\nabla}_j N + NK_i{}^k K_{kj} - \mathcal{L}ie_{\vec{N}}K_{ij}\right)$$

$$\mathcal{L}ie_{\vec{N}}K_{ij} \equiv N^k\partial_k K_{ij} + K_{ik}\partial_j N^k + K_{jk}\partial_i N^k$$

Computing the Riemann components 3

- Finally, in the standard GR we use $g^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \gamma^{-1} \end{pmatrix} - n \times n^T$

- and, using $\gamma^{ij}\dot{K}_{ij} = \partial_0 K^i_i + K^{ij}\dot{\gamma}_{ij}$ and trading velocities for extrinsic curvatures again, we get

$$
\begin{aligned}
R = g^{\mu\nu} g^{\alpha\beta} R_{\mu\alpha\nu\beta} &= \gamma^{ik}\gamma^{jl} R_{ijkl} - 2n^\mu n^\nu \gamma^{ij} R_{\mu i\nu j} \\
&= \overset{(3)}{R} + K^i_i K^j_j - 3K^{ij}K_{ij} - \frac{2}{N}\gamma^{ij}\dot{K}_{ij} + \frac{4}{N}K^{ij}\overset{(3)}{\nabla}_j N_i + 2\frac{N^j}{N}\overset{(3)}{\nabla}_j K^i_i - \frac{2}{N}\overset{(3)}{\triangle} N \\
&= \overset{(3)}{R} + K^{ij}K_{ij} + K^i_i K^j_j - \frac{2}{N}\dot{K}^i_i + 2\frac{N^j}{N}\overset{(3)}{\nabla}_j K^i_i - \frac{2}{N}\overset{(3)}{\triangle} N
\end{aligned}
$$

For the Einstein-Hilbert density, we use $\sqrt{-g} = N\sqrt{\gamma}$, $\partial_0\sqrt{\gamma} = \frac{\sqrt{\gamma}}{2}\gamma^{ij}\dot{\gamma}_{ij}$ and get

$$
\sqrt{-g}R = \sqrt{\gamma}N\left(\overset{(3)}{R} + K^{ij}K_{ij} - K^i_i K^j_j\right) - 2\sqrt{-g}\,\nabla_\mu\left(K^i_i n^\mu\right) - 2\sqrt{\gamma}\overset{(3)}{\triangle} N
$$

where $-2\sqrt{-g}\,\nabla_\mu\left(K^i_i n^\mu\right) \equiv -2\partial_0\left(\sqrt{\gamma}K^i_i\right) + 2\sqrt{\gamma}\overset{(3)}{\nabla}_j\left(K^i_i N^j\right)$.

However, we have all the Riemann tensor components at hand, and more general models can be studied.

Hassan-Rosen analysis 1

- In dRGT gravity one has to work with the square root of

$$g^{\mu\alpha}\eta_{\alpha\nu} = \begin{pmatrix} \frac{1}{N^2} & \frac{N^i}{N^2} \\ -\frac{N^j}{N^2} & \gamma^{ij} - \frac{N^i N^j}{N^2} \end{pmatrix}$$

where we have assumed the Minkowski auxiliary metric $f = \eta$

- The approach of Hassan and Rosen is to take the root
- It would have been fairly simple if it were not for the γ since

$$\begin{pmatrix} 1 & a^i \\ -a^j & -a^i a^j \end{pmatrix}^2 = \left(1 - a^k a^k\right) \begin{pmatrix} 1 & a^i \\ -a^j & -a^i a^j \end{pmatrix}$$

- How to proceed with the actual case at hand?

Hassan, Rosen; 1106.3344 1109.3515 1111.2070

Hassan-Rosen analysis 2

- The main idea is to make a clever redefinition of shits

$$N_i = \left(\delta_i^j + N D_i^j(n, \gamma)\right) n_j$$

such that

$$\sqrt{g^{-1}\eta} = \frac{1}{N\sqrt{1 - n^k n^k}} \begin{pmatrix} 1 & n^i \\ -n^j & -n^i n^j \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & X^{ij}(\gamma, n) \end{pmatrix}$$

- We see that, after this redefinition, the quantity $\sqrt{-g}\,V$ in the action is linear in the lapse, and therefore one of the constraints remains, leaving 5 degrees of freedom
- One can show that the secondary constraint is also there (it is a second class couple), and the analysis can be extended to arbitrary background metrics, and to bimetric versions too.
- In higher potentials the $\frac{1}{N^2}$ and $\frac{1}{N^3}$ terms drop out due to specific properties of the first matrix in the expression for $\sqrt{g^{-1}f}$

? Some particular type of the square root is (partially) chosen ?

Matrix square roots...

- What does it mean to have a matrix square root?
- One can use interpolation polynomials, or definitions in terms of Jordan blocks – functional calculus on the algebra of matrices.
- Unique answers, usual smoothness, but better understood on the field of complex numbers
- We can also take a naive understanding of square roots.
- For that, one can introduce the square root implicitly
  Golovnev, Phys.Lett.B **707** (2012), 404-408

$$N\,V = 2\mathrm{m}^2\,\Phi^{\mu}_{\mu} + \kappa^{\mu}_{\nu}\left(\Phi^{\nu}_{\alpha}\Phi^{\alpha}_{\mu} - N^2 g^{\nu\alpha}\eta_{\alpha\mu}\right)$$

or, after integrating out the Lagrange multiplier,

$$N\,V = m^2\left(\Phi^{\mu}_{\mu} + \left(\Phi^{-1}\right)^{\mu}_{\nu} N^2 g^{\nu\alpha}\eta_{\alpha\mu}\right)$$

- Now the constraint analysis can be done without explicitly taking any square roots; and it was used in Stueckelberg formalism (Hassan et al.).

Functional calculus

$$J_k = \begin{bmatrix} \lambda_k & 1 & & \\ & \lambda_k & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_k \end{bmatrix} \in \mathbb{C}^{m_k \times m_k},$$

$$f(J_k) = \begin{bmatrix} f(\lambda_k) & f'(\lambda_k) & \dots & \frac{f^{(m_k-1)}(\lambda_k)}{(m_k-1)!} \\ & f(\lambda_k) & \ddots & \vdots \\ & & \ddots & f'(\lambda_k) \\ & & & f(\lambda_k) \end{bmatrix}.$$

For a two-dimensional $-I$ matrix we get iI or $-iI$, although there is a real root – the standard symplectic structure.

Vielbeins in ADM

- There is also a very interesting vierbein (vielbein) formulation of the model (Hinterbichler, Rosen; 1203.5783). Moreover, this approach allows to tackle the multi-metric theories. The ghost free potentials are antisymmetrised wedge products of the vielbeins in the model – a very nice and beautiful result!
- After a proper rotation in the tangent space, a connection can be brought to the upper-triangular form. And then, one easily determines the ADM-like parametrisation

$$\hat{E}_\mu{}^A = \begin{pmatrix} N & N^i e_i{}^a \\ 0 & e_i{}^a \end{pmatrix}$$

- Due to antisymmetrisation, the lapse will go linearly in the action

Vielbein formulation of multimetric interactions

- It is natural that the square-root structures admit more natural representation in terms of vielbeins (Hinterbichler, Rosen 2012). For each metric we have the standard Einstein-Hilbert $\int R \wedge e \wedge e$
- Each metric has its representative vielbein $e_{(i)}$ and the interaction terms are

$$\int \varepsilon_{a_1 \dots a_D} e_{(1)}^{a_1} \wedge \dots \wedge e_{(1)}^{a_{i_1}} \wedge e_{(2)}^{a_{i_1+1}} \wedge \dots \wedge e_{(2)}^{a_{i_1+i_2}} \wedge \dots \wedge e_{(n)}^{a_D}$$

- where k-th vielbein enters i_k times, and the total number of factors equals the dimensionality of space-time, D.
- It was shown that transition to metric formulation requires a symmetric vielbein condition, $e_{\mu a} = e_{a\mu}$ for massive gravity with Minkowski reference metric (Deffayet et al. 2012).
- Recently, it was claimed that this (Deser-van Nieuwenhuizen) condition follows from the vielbein action as an equation of motion for the Lorentz-symmetry Stueckelberg field (Ondo, Tolley 2013).
- The metric formulation with more than two interacting metrics seems to be tricky.

Another class of bimetric theories

- Recently, the actions of the form $\int \sqrt{-g}\, g^{\mu\nu} R_{\mu\nu}(\hat{g})$ have been proposed (Amendola, Enqvist, Koivisto; 1010.4776)
- Unfortunately, the ghosts appear. For fluctuations of g (denote them by h) and the difference between the two metrics (denote it by f) we have at the quadratic level EH(h) – EH(f). (Beltran Jimenez, Golovnev, Karciauskas, Koivisto; 1201.4018) Not very surprising! Consider $\int dt \left(\ddot{x}+\dot{x}^2\right) y$
- At non-linear level, there are also dynamical vectors and scalars. The conformal mode of the metric difference was shown to be healthy (Koivisto; 1103.2743)
- The tensor ghosts might be cured by combining the new term with the standard GR (1201.4018), but then it flips the sign of conformal mode too.

  The ghosts are ubiquitous!
- Can we use some $f\left(g^{\mu\nu} R_{\mu\nu}(\hat{g}),\, g^{\mu\alpha} g^{\nu\beta} R_{\mu\nu}(\hat{g}) R_{\alpha\beta}(\hat{g})\right)$ model? Hard question in general. May be, one might try to look for a function with one degree of freedom less? ADM analysis might again be useful!

Other models: computing the Ricci

- For computing the Ricci tensor components it is still good to use the unit normal vector instead of taking the time components:

$$R_{ij} = \overset{(3)}{R}_{ij} - \frac{1}{N}\dot{K}_{ij} - \frac{1}{N}\overset{(3)}{\nabla}_i\overset{(3)}{\nabla}_j N + K_{ij}K^k_k - 2K_{ik}K^k_j + \frac{1}{N}\mathcal{L}ie_{\vec{N}}K_{ij}$$

$$n^\mu R_{\mu i} = \overset{(3)}{\nabla}_i K^j_j - \overset{(3)}{\nabla}_j K^j_i$$

$$n^\mu n^\nu R_{\mu\nu} = \frac{1}{N}\gamma^{ij}\dot{K}_{ij} + \frac{1}{N}\overset{(3)}{\triangle}N + K_{ij}K^{ij} - \frac{1}{N}\gamma^{ij}\mathcal{L}ie_{\vec{N}}K_{ij}$$

- For an action of the form $\int \sqrt{-\hat{g}}\,\hat{g}^{\mu\nu} R_{\mu\nu}(g)$ we can express everything in terms of the above quantities for g and the difference of the two metrics.
- We can straightforwardly obtain a combination of the form $\chi^{ij}\dot{K}_{ij}$ which shows that an independent combination of metrics χ^{ij} acquires dynamics with unbounded kinetic function.

Non-metric connections

- It is more comfortable to calculate Riemann tensor components for the metric connection due to its symmetries. After that, any connection might be presented as a sum of the metric part and the variation which, in turn, consists of the contortion tensor (to represent the effects of torsion) and the contribution from the non-metricity tensor. And exact relation for the Riemann tensor would take the form

$$\delta R = \nabla \delta\Gamma - \nabla \delta\Gamma + \delta\Gamma \cdot \delta\Gamma - \delta\Gamma \cdot \delta\Gamma$$

- Suppose the connection is generated by some modification of the physical metric

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \bar{h}_{\mu\nu}$$

- In this case we get

$$\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}\bar{g}^{\mu\nu}\left(\nabla_\alpha \bar{h}_{\nu\beta} + \nabla_\beta \bar{h}_{\nu\alpha} - \nabla_\nu \bar{h}_{\alpha\beta}\right)$$

- The inverse metric can be computed in ADM-type variables, or in perturbation theory:

$$\bar{g}^{\mu\nu} = \left[\left(g + \bar{h}\right)^{-1}\right]^{\mu\nu} = \left[\left(\mathbb{I} + g^{-1}\bar{h}\right)^{-1}\right]^{\mu}_{\alpha} g^{\alpha\nu} = \left[\sum_{n=0}^{\infty}\left(-g^{-1}\bar{h}\right)^{n}\right]^{\mu}_{\alpha} g^{\alpha\nu}$$

Perspectives

- The ADM type of analysis can be successfully applied to modified gravity theories. It gives very powerful tools for understanding a model at hand, e.g. to count the degrees of freedom in a completely non-perturbative manner.
- Beyond the perturbative approach, the dRGT massive gravity exhibits some problems with the existence and uniqueness of the square root matrix
- It is interesting to further investigate the relations among the metric formulation, the vielbein approach, auxiliary fields method, etc.
- At the same time, other modifications and bimetric models are worth studying
- In the simplest version of the metric-affine Amendola-Enqvist-Koivisto theory, there are ghosts around the (double) Minkowski background
- One way to study more general set-ups of this class is to perform a suitable ADM-type analysis
- It seems feasible to do so.....